JOURNAL OF MATERIALS SCIENCE 21 (1986) 2759–2762

A theory of acoustic measurement of the elastic constants of a general anisotropic solid

W. C. VAN BUSKIRK, S. C. COWIN, R. CARTER Jr
Department of Biomedical Engineering, Tulane University, New Orleans, Louisiana 70118, USA

An acoustic wave approach is presented for the measurement of the twenty-one independent elastic constants of the most general linearly elastic anisotropic solid. The method requires that one be able to measure the density of the material, the velocities of the three modes of wave propagation in each of six directions, and the particle displacements associated with each of those modes.

1. Introduction

Generally, the measurement of the anisotropic elastic constants of a material begins with an assumption that the axes of symmetry and/or the planes of reflective symmetry of the material are known. In the case of a pure crystal this is a reasonable assumption. It is a more speculative assumption in the case where the material is not a pure crystal, for example if the material is a common geological, biological or construction material. The generalized Hooke's law relates the stress tensor T_{ij} to the strain tensor E_{km}

$$T_{ij} = C_{ijkm}E_{km} \tag{1}$$

where C_{ijkm} are the components of the elasticity tensor. In general there are twenty-one distinct components of the elasticity tensor. Although for specific anisotropic material symmetries relative to specific co-ordinate systems there are less than twenty-one distinct and non-zero components of C_{ijkm}, in general, for most anisotropic material symmetries relative to arbitrarily selected co-ordinate systems there are twenty-one distinct and non-zero components of C_{ijkm}. The representations of the elasticity tensor C_{ijkm} for specific material symmetries, worked out by Voigt in the last century and described in many textbooks [1, 2], only hold for co-ordinate systems peculiar to the symmetry being represented. These co-ordinate systems are determined by the axes of symmetry and planes of reflective symmetry that describe a particular material symmetry.

If the particular type of material anisotropy of a specimen is unknown, then relative to an arbitrarily selected co-ordinate system the material must be assumed to have twenty-one distinct and non-zero components of C_{ijkm}. Cowin and Mehrabadi [3] have recently presented a method of identifying elastic material symmetries if one is given the components of C_{ijkm} relative to an arbitrary co-ordinate system. The problem of material symmetry identification is thus reduced to the measurement of the twenty-one components. Hayes [4] described a series of static mechanical tests to measure the twenty-one components. However, traditional mechanical test methods are difficult to apply to a material that is both anisotropic and heterogeneous, and many materials are both anisotropic and heterogeneous. Heterogeneity requires that the test specimens be small in order to insure that the properties are nearly uniform throughout the test specimen. Anisotropy requires that traditional mechanical tests be applied in several different directions in order to obtain enough information for the calculation of all the components of C_{ijkm}, as shown by Hayes [4]. Since it is not possible to fabricate many specimens from the same portion of material, the combined effects of heterogeneity and anisotropy make it impossible to measure anisotropic elastic properties by traditional mechanical testing methods.

The problems induced in the measurement of elastic coefficients by anisotropy and heterogeneity, and the need for multiple specimens, can be eliminated by using test procedures involving elastic waves of ultrasonic frequency. Van Buskirk *et al.* [5] describe a procedure in which all nine orthotropic elastic constants of bone are determined from a single 5 mm cubical specimen. The method consists of assuming orthotropic symmetry and the orientation of the axes of symmetry on the basis of visual inspection, then determining the longitudinal and two shear wave velocities along each of the three orthotropic symmetry axes. Along the 45° axis bisecting the angle between each of the orthotropic symmetry axes the quasi-longitudinal and quasi-transverse wave velocities are determined. These 15 wave velocities and the density of the specimen can be used to determine the nine orthotropic elastic constants and some internal checks on the consistency of the values of the constants. Van Buskirk *et al.* [5] used a pulse transmission method to determine wave velocity. Ashman *et al.* [6] introduced a continuous wave technique for the measurement of wave velocity and document the anisotropy and heterogeneity of the bone-tissue in the human and canine femur. Recently Ashman *et al.* [7] employed the continuous wave method to show that the cortical bone tissue of the canine mandible was elastically isotropic and homogeneous. They assumed that the material was orthotropically elastic and heterogeneous and employed the statistical method known as two factor analysis of variance with repeated measures on the experimentally determined coefficients to demonstrate the elastic isotropy and homogeneity.

The experimental method consisting of the passage of waves of ultrasonic frequency in many directions through the same specimen can be extended to the determination of the twenty-one components C_{ijkm} relative to an arbitrary, but known, co-ordinate system, if one assumes that the direction of particle displacement as well as the wave speed associated with the three principal waves propagating in a particular material direction can be measured. In this paper we present the theoretical developments which relate the twenty-one components of C_{ijkm} to the density of the material and the three principal wave speeds and directions of particle displacement in six different directions through a material specimen.

Elastic wave propagation is reviewed briefly in the next section and formulae relating the components of C_{ijkm} to the acoustic wave measurements are developed in the section following that. In the final section we sketch an experimental protocol for the implementation of the formulae developed.

2. Elastic wave propagation

In the absence of body forces the equation of motion for a continuum body is

$$T_{ij,j} = \varrho \ddot{u}_i \quad (2)$$

where T_{ij} is a component of the stress tensor relative to a Cartesian basis we denote by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. ϱ is the density, and u_i is a component of the displacement vector. The comma represents partial differentiation and the dot denotes a derivative with respect to time. The usual summation convention is employed. Equation 1 is the constitutive equation for a linearly elastic anisotropic material. The relationship between the strain tensor and displacement is given by

$$E_{km} = \tfrac{1}{2}(u_{k,m} + u_{m,k}) \quad (3)$$

If we introduce Equations 2 and 3 into Equation 1 and employ the symmetry properties of the elasticity tensor we obtain

$$C_{ijkm} u_{k,mj} = \varrho \ddot{u}_i \quad (4)$$

If now we assume a plane wave propagating in the direction $\boldsymbol{n}$ with components n_i, the displacement is

$$u_i = a_i e^{(n_j x_j - vt)} \quad (5)$$

where a_i is a component of the amplitude of particle oscillation, x_i is a component of the particle position, v is the speed of wave propagation, and t is time. If we introduce Equation 5 into Equation 4 we obtain

$$a_k C_{ijkm} n_j n_m = \varrho v^2 a_i \quad (6)$$

We now introduce the notation

$$\Gamma_{ik} = C_{ijkm} n_j n_m \quad (7)$$

where Γ_{ik} is a component of the matrix known as the Kelvin–Christoffel stiffness matrix. Using this notation in Equation 6 we see that

$$\Gamma_{ik} a_k = \varrho v^2 a_i \quad (8)$$

If for a tensor L_{ij}, we have the relationship $L_{ij} m_j = \lambda m_i$, then m_i is a component of an eigenvector of the tensor and λ is the corresponding eigenvalue. Thus, a_i is a component of an eigenvector of Γ_{ik} and ϱv^2 is the corresponding eigenvalue.

Γ_{ik} is a real symmetric matrix. Therefore for a wave propagating in any direction in a general anisotropic material there are three modes of propagation, each with a different velocity and a different direction of particle displacement. The directions of particle displacement for the three modes are given by the eigenvectors of Γ_{ik} and are orthogonal. The velocities of the three modes are related to the eigenvalues, λ_j, of Γ_{ik} through the equation

$$v_j = (\lambda_j/\varrho)^{1/2} \quad (9)$$

We denote the three components of the eigenvector associated with λ_j by a_{1j}, a_{2j} and a_{3j}.

As is customary in the discussion of linearly elastic anisotropic materials we introduce a single index notation for stress and strain as shown in Table I. In this notation the constitutive equation $T_{ij} = C_{ijkm} E_{km}$ becomes $\sigma_i = c_{ij}\varepsilon_j$ where the summation convention is employed with a range of six and c_{ij} is known as the stiffness matrix. The twenty-one independent elastic constants are the components of the symmetric stiffness matrix. The components of the Kelvin–Christoffel stiffness matrix in terms of the components of the stiffness matrix, c_{ij}, are shown in Table II.

To summarize, if one knows the stiffness matrix for a material it is possible to determine the components of the Kelvin–Christoffel stiffness matrix for any direction $\boldsymbol{n}$ with components n_i. Using the Kelvin–Christoffel stiffness matrix for a particular direction, it is possible to determine the velocities and particle displacements for the three modes of wave propagation in that direction. The converse is also possible as we will show below. That is to say, one can determine the components of the Kelvin–Christoffel stiffness matrix

TABLE I

Stresses	Strains
$\sigma_1 = T_{11}$	$\varepsilon_1 = E_{11}$
$\sigma_2 = T_{22}$	$\varepsilon_2 = E_{22}$
$\sigma_3 = T_{33}$	$\varepsilon_3 = E_{33}$
$\sigma_4 = T_{23} = T_{32}$	$\varepsilon_4 = 2E_{23} = 2E_{32}$
$\sigma_5 = T_{13} = T_{31}$	$\varepsilon_5 = 2E_{13} = 2E_{31}$
$\sigma_6 = T_{23} = T_{32}$	$\varepsilon_6 = 2E_{12} = 2E_{21}$

TABLE II

$$\Gamma_{11} = n_1^2 c_{11} + n_2^2 c_{66} + n_3^2 c_{55} + 2n_2 n_3 c_{56} + 2n_3 n_1 c_{15} + 2n_1 n_2 c_{16}$$

$$\Gamma_{22} = n_1^2 c_{66} + n_2^2 c_{22} + n_3^2 c_{44} + 2n_2 n_3 c_{24} + 2n_3 n_1 c_{46} + 2n_1 n_2 c_{26}$$

$$\Gamma_{33} = n_1^2 c_{55} + n_2^2 c_{44} + n_3^2 c_{33} + 2n_2 n_3 c_{34} + 2n_3 n_1 c_{35} + 2n_1 n_2 c_{45}$$

$$\Gamma_{12} = \Gamma_{21} = n_1^2 c_{16} + n_2^2 c_{26} + n_3^2 c_{45} + n_2 n_3 (c_{46} + c_{25}) + n_3 n_1 (c_{14} + c_{56}) + n_1 n_2 (c_{12} + c_{66})$$

$$\Gamma_{13} = \Gamma_{31} = n_1^2 c_{15} + n_2^2 c_{46} + n_3^2 c_{35} + n_2 n_3 (c_{45} + c_{36}) + n_3 n_1 (c_{13} + c_{55}) + n_1 n_2 (c_{14} + c_{56})$$

$$\Gamma_{23} = \Gamma_{32} = n_1^2 c_{56} + n_2^2 c_{24} + n_3^2 c_{34} + n_2 n_3 (c_{44} + c_{23}) + n_3 n_1 (c_{36} + c_{45}) + n_1 n_2 (c_{25} + c_{46})$$

TABLE III

$\Gamma_{11} = a_{11}^2 \varrho v_1^2 + a_{12}^2 \varrho v_2^2 + a_{13}^2 \varrho v_3^2$	
$\Gamma_{22} = a_{21}^2 \varrho v_1^2 + a_{22}^2 \varrho v_2^2 + a_{23}^2 \varrho v_3^2$	
$\Gamma_{33} = a_{31}^2 \varrho v_1^2 + a_{32}^2 \varrho v_2^2 + a_{33}^2 \varrho v_3^2$	
$\Gamma_{12} = a_{11} a_{21} \varrho v_1^2 + a_{12} a_{22} \varrho v_2^2 + a_{13} a_{23} \varrho v_3^2$	
$\Gamma_{13} = a_{11} a_{31} \varrho v_1^2 + a_{12} a_{32} \varrho v_2^2 + a_{13} a_{33} \varrho v_3^2$	
$\Gamma_{23} = a_{21} a^{32} \varrho v_1^2 + a_{22} a_{32} \varrho v_2^2 + a_{23} a_{33} \varrho v_3^2$	

for a particular direction from a knowledge of the velocities and particle displacements for the three modes of wave propagation in that direction. If one knows the components of six Kelvin–Christoffel stiffness matrices, it is possible to determine all of the components of the stiffness matrix for the material.

3. Determination of the elasticity tensor from elastic wave data

Consider a rectangular co-ordinate system with unit basis vectors $\boldsymbol{e}_i'$ related to $\boldsymbol{e}_i$ by an orthogonal tensor Q through the equations

$$\boldsymbol{e}_i' = Q_{mi} \boldsymbol{e}_m' \quad (10)$$

The components of the Kelvin–Christoffel stiffness matrix for a direction $\boldsymbol{n}$ with respect to $\boldsymbol{e}_i'$ are given by the well known tensor transformation law

$$\Gamma_{ij}' = Q_{mi} Q_{nj} \Gamma_{mn}. \quad (11)$$

Also, since the transformation is orthogonal

$$\Gamma_{mn} = Q_{mi} Q_{nj} \Gamma_{ij}'. \quad (12)$$

If the basis vectors $\boldsymbol{e}_i'$ are the eigenvectors of the Kelvin–Christoffel stiffness matrix, then

$$[\Gamma_{ij}'] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \varrho v_1^2 & 0 & 0 \\ 0 & \varrho v_2^2 & 0 \\ 0 & 0 & \varrho v_3^2 \end{bmatrix} \quad (13)$$

and $Q_{ij} = a_{ij}$, the components of the eigenvectors. Thus we can write

$$\Gamma_{mn} = \boldsymbol{a}_{mi} a_{nj} \Gamma_{ij}'. \quad (14)$$

Therefore, if we know the density of the material, the velocities of the three modes of propagation in a particular direction, and the particle displacement associated with each of those modes, we can determine each of the components of the Kelvin–Christoffel stiffness matrix for that direction. The individual formulae are given in Table III.

We now proceed to show how this approach can be used to determine the twenty-one independent elastic constants of a general anisotropic material. Consider

TABLE IV

A	Direction of propagation
1	$\boldsymbol{n} = \boldsymbol{e}_1$
2	$\boldsymbol{n} = \boldsymbol{e}_2$
3	$\boldsymbol{n} = \boldsymbol{e}_3$
4	$\boldsymbol{n} = 2^{-1/2}(\boldsymbol{e}_2 + \boldsymbol{e}_3)$
5	$\boldsymbol{n} = 2^{-1/2}(\boldsymbol{e}_1 + \boldsymbol{e}_3)$
6	$\boldsymbol{n} = 2^{-1/2}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$

TABLE V

$c_{11} = \Gamma_{11}(1)$*	$c_{22} = \Gamma_{22}(2)$	$c_{33} = \Gamma_{33}(3)$
	$c_{44} = \Gamma_{33}(2) = \Gamma_{22}(3)$	
	$c_{55} = \Gamma_{33}(1) = \Gamma_{11}(3)$	
	$c_{66} = \Gamma_{22}(1) = \Gamma_{11}(2)$	
$c_{15} = \Gamma_{13}(1)$	$c_{16} = \Gamma_{12}(1)$	$c_{56} = \Gamma_{23}(1)$
$c_{24} = \Gamma_{23}(2)$	$c_{26} = \Gamma_{12}(2)$	$c_{46} = \Gamma_{13}(2)$
$c_{34} = \Gamma_{23}(3)$	$c_{35} = \Gamma_{13}(3)$	$c_{45} = \Gamma_{12}(3)$
	$c_{12} = 2\Gamma_{12}(6) - c_{16} - c_{26} - c_{66}$	
	$c_{13} = 2\Gamma_{13}(5) - c_{15} - c_{35} - c_{55}$	
	$c_{23} = 2\Gamma_{23}(4) - c_{24} - c_{34} - c_{44}$	
$c_{14} = 2\Gamma_{13}(6) - c_{15} - c_{46} - c_{56} = 2\Gamma_{12}(5) - c_{16} - c_{45} - c_{56}$		
$c_{25} = 2\Gamma_{23}(6) - c_{24} - c_{46} - c_{56} = 2\Gamma_{12}(4) - c_{26} - c_{46} - c_{45}$		
$c_{36} = 2\Gamma_{23}(5) - c_{34} - c_{45} - c_{56} = 2\Gamma_{13}(4) - c_{35} - c_{45} - c_{46}$		

*Number in bracket refers to A-direction 1 to 6.

a wave in the direction $\boldsymbol{n}_1 = \boldsymbol{e}_1$ (i.e. $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$). From Table II we see that $\boldsymbol{\Gamma}_{11} = c_{11}$ for this direction. From Table III we see that

$$\boldsymbol{\Gamma}_{11} = c_{11} = a_{11}^2 \varrho v_1^2 + a_{12}^2 \varrho v_2^2 + a_{13}^2 \varrho v_3^2$$

Thus, if we can measure the velocities and particle displacements for the three modes of a wave propagating in the $\boldsymbol{e}_1$-direction, we have the information we need to determine c_{11}.

As another example consider a wave in the direction $\boldsymbol{n} = \frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$ (i.e. $n_1 = 2^{-1/2}$, $n_2 = 2^{-1/2}$, and $n_3 = 0$). From Table II we see that $\Gamma_{12} = \frac{1}{2}(c_{12} + c_{16} + c_{26} + c_{66})$ or $c_{12} = 2\Gamma_{12} - c_{16} - c_{26} - c_{66}$. We can find Γ_{12} using the formula in Table III, and it is possible to determine c_{16}, c_{26} and c_{66} directly from other experiments. Thus, we can find c_{12}.

To measure all twenty-one elastic constants it is necessary to determine the velocities and particle displacements of waves propagating in six different directions in a material. We denote these directions by $A = 1, 2, 3, 4, 5$ and 6. Six convenient directions are shown in Table IV.

Using the designation $\Gamma_{ij}(A)$ to represent a component of the Kelvin–Christoffel stiffness matrix associated with the A-direction, formulae for the twenty-one independent elastic constants of the stiffness matrix are given in Table V. Formulae for $\Gamma_{ij}(A)$ are given in Table III in terms of appropriate mode velocities and eigenvector components.

4. Remarks on experimental method

In this section we discuss how the theory might be used to make actual measurements. A specimen in the form of a cube is cut from the material. A Cartesian co-ordinate system with axes x_1, x_2 and x_3 parallel to the edges of the cube is designated. Consider first wave propagation in the x_1-direction. We do not know *a priori* the directions of particle oscillation for the three modes of propagation in that direction. Thus, it is necessary to generate a variety of waves with dominant particle oscillations in different directions. Ultrasound waves of this nature can be generated using commercially available piezoelectric transducers. The arrival of the wave at the other face perpendicular to the x_1-axis is sensed by another transducer or set of transducers. Transducers are available which are sensitive to longitudinal oscillations and a shear oscillations. A longitudinal transducer would pick up the

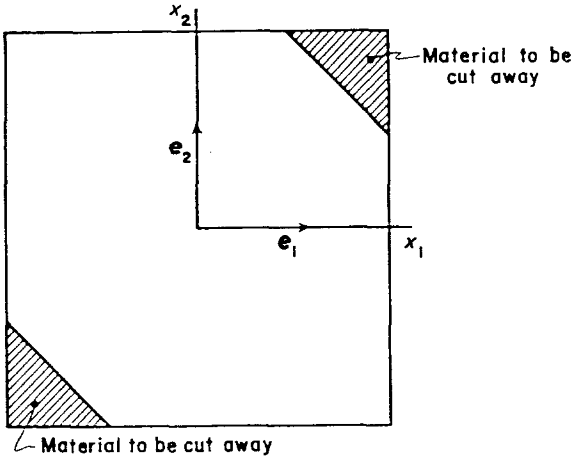

Figure 1 An illustration of the material of the specimen that is to be cut away.

longitudinal component of the wave and a shear transducer could be used to find the transverse components. Shear transducers are sensitive to oscillations in a particular direction. Thus, it is necessary to be able to rotate the shear transducer through 180° to find the dominant transverse components. Once the three modes of particle oscillation for a propagating wave have been determined, waves are generated in which each of these is the dominant one. Wave speeds are determined using these "dominant mode" waves.

Once the wave speeds and particle oscillations for the three modes of wave propagation in the x_1-direction have been determined, the measurement is repeated in the x_2- and x_3-directions. Using the information gathered in these three measurements, it is possible to determine the fifteen elastic constants in the top three groups in Table V.

The specimen is then cut on the edge of the surfaces perpendicular to the x_1- and x_2-directions such as to form two faces with unit normals $\boldsymbol{n} = \pm 2^{-1/2}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$ as shown in Fig. 1. The basic measurement is repeated for waves propagating between these two faces. Two more sets of cuts need to be made on the specimen. One set produces faces with unit normals $\boldsymbol{n} = \pm 2^{-1/2}$ $(\boldsymbol{e}_1 + \boldsymbol{e}_3)$ and the other set produces faces with unit normals $\boldsymbol{n} = \pm 2^{-1/2}(\boldsymbol{e}_2 + \boldsymbol{e}_3)$. The appearance of the specimen after the cuts have been made is shown in Fig. 2. The basic measurement is again performed in these two directions. Using nine of the fifteen elastic constants found in the first three measurements and the information found in the last three measurements, we can determine the remaining six elastic constants of the stiffness matrix.

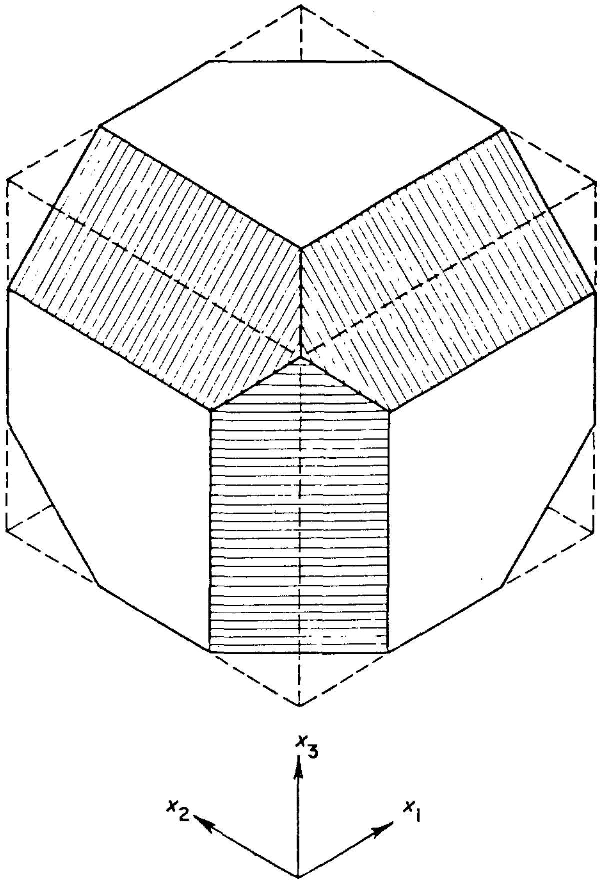

Figure 2 A sketch of the appearance of the specimen after all the cuts have been made.

Acknowledgement

The work was supported by grants from the NIADDK and NIGMS, specifically from the Musculoskeletal Program of NIADDK and the Marc Program of NIGMS.

References

1. W. VOIGT, in "Lehrbuch der Kristall Physik" (Teubner, Leipzig, 1910) p. 560.
2. A. E. H. LOVE, in "Treatise on the Mathematical Theory of Elasticity" (Dover, New York, 1927) p. 157.
3. S. C. COWIN and M. M. MEHRABADI, *Quart. J. Appl. Mech. Math*, in press.
4. M. HAYES, *J. Mater. Sci.* **4** (1969) 10.
5. W. C. VAN BUSKIRK, S. C. COWIN and R. N. WARD, *J. Biomech. Eng.* **103** (1981) 67.
6. R. B. ASHMAN, S. C. COWIN, W. C. VAN BUSKIRK and J. C. RICE, *J. Biomechanics* **17** (1984) 349.
7. R. B. ASHMAN, G. ROSINIA, S. C. COWIN, M. G. FONTENOT and J. C. RICE, *ibid.* **18** (1985) 717.

Received 19 August
and accepted 3 October 1985